

Past and Future of Cosmic Topology

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Abstract

The global topology of the universe, including questions about the shape of space, its volume and its connectedness, is a fundamental issue in cosmology which has been overlooked for many years, except by some pioneering authors (see references in Lachièze-Rey & Luminet, 1995, hereafter LaLu95). In the first part of the present article, I set out some unexplored historical material about the early development of cosmic topology. It stems out that the two “fathers” of the big bang concept, Friedmann and Lemaître, were also the first to realize the full importance of cosmic topology, whereas Einstein remained reluctant to the idea of a multi-connected space. In the second part I briefly comment new developments in the field since 1995, both from a theoretical and an observational point of view. They fully confirm that cosmic topology is, more than ever, a promising field of investigation.

1 Birth of cosmic topology

One of the oldest cosmological questions is the physical extension of space : is it finite or infinite? (see e.g. Luminet, 1994 ; Luminet & Lachièze-Rey, 1994). In the history of cosmology, it is well known that the Newtonian physical space, mathematically identified with infinite Euclidean space \mathbf{R}^3 , gave rise to paradoxes such as darkness of night (see e.g. Harrison, 1987) and to problems of boundary conditions. Regarding for instance the Mach’s idea according to which local inertia would result from the contributions of masses at infinity, an obvious divergence difficulty arose, since a homogeneous Newtonian universe with non-zero density had an infinite mass.

The aim of relativistic cosmology was to deduce from gravitational field equations some physical models of the universe as a whole. When Einstein (1917) assumed in his static cosmological solution that space was a positively-curved hypersphere, one of his strongest motivations was to provide a model for a *finite* space, although without a boundary. He regarded the closure of space as necessary to solve the problem of inertia (Einstein, 1934). The spherical model cleared up most of the paradoxes stemmed from Newtonian cosmology in such an elegant way that most cosmologists of the time adopted the new paradigm of a closed space, without examining other geometrical possibilities. Einstein was also convinced that the hypersphere provided not only the metric of cosmic

space – namely its local geometrical properties – but also its global structure, namely its topology. However, topology does not seem to have been a major preoccupation of Einstein; his 1917 cosmological article did not mention any topological alternative to the spherical space model.

Some of his colleagues pointed out to Einstein the arbitrariness of his choice. The reason was the following. The global shape of space is not only depending on the metric; it primarily depends on its topology, and requires a complementary approach to Riemannian differential geometry. Since Einstein's equations are partial derivative equations, they describe only local geometrical properties of spacetime. The latter are contained in the metric tensor, which enables us to calculate the components of the curvature tensor at any non-singular point of spacetime. But Einstein's equations do not fix the global structure of spacetime : to a given metric solution of the field equations, correspond several (and in most cases an infinite number of) topologically distinct universe models.

First, De Sitter (1917) noticed that the Einstein's solution admitted a different space-form, namely the 3-dimensional projective space (also called elliptic space), obtained from the hypersphere by identifying antipodal points. The projective space has the same metric as the spherical one, but a different topology (for instance, for the same curvature radius its volume is twice smaller).

Next H. Weyl pointed out the freedom of choice between spherical and elliptical topologies. The Einstein's answer (1918) was unequivocal : “Nevertheless I have like an obscure feeling which leads me to prefer the spherical model. I have the presentiment that manifolds in which any closed curve can be continuously contracted to a point are the simplest ones. Other persons must share this feeling, otherwise astronomers would have taken into account the case where our space is Euclidean and finite. Then the two-dimensional Euclidean space would have the connectivity properties of a ring surface. It is an Euclidean plane in which any phenomenon is doubly periodic, where points located in the same periodical grid are identical. In finite Euclidean space, three classes of non continuously contractible loops would exist. In a similar way, the elliptical space possesses a class of non continuously contractible loops, contrary to the spherical case; it is the reason why I like it less than the spherical space. Can it be proved that elliptical space is the only variant of spherical space? It seems yes to me”.

Einstein (1919) repeated his argumentation in a postcard sent to Felix Klein : “I would like to give you a reason why the spherical case should be preferred to the elliptical case. In spherical space, any closed curve can be continuously contracted to a point, but not in the elliptical space; in other words the spherical space alone is simply-connected, not the elliptical one [...] Finite spaces of arbitrary volume with the Euclidean metric element undoubtedly exist, which can be obtained from infinite spaces by assuming a triple periodicity, namely identity between sets of points. However such possibilities, which are not taken into account by general relativity, have the wrong property to be multiply-connected”. From these remarks it follows that the Einstein's prejudice in favour of simple-connectedness of space was of an aesthetical nature, rather than being based on physical reasoning.

In his answer to Weyl, Einstein was definitely wrong on the last point : in dimension three, an infinite number of topological variants of the spherical space – all closed – do exist, including the so-called lens spaces (whereas in dimension two, only two spherical

spaceforms exist, the ordinary sphere and the elliptic plane). However, nobody knew this result in the 1920's : the topological classification of 3-dimensional spaces was still at its beginnings. The study of Euclidean spaceforms started in the context of crystallography. Feodoroff (1885) classified the 18 symmetry groups of crystalline structures in \mathbf{R}^3 , Bieberbach (1911) developed a full theory of crystallographic groups, and twenty years later only Nowacki (1934) showed how the Bieberbach's results could be applied to achieve the classification of 3-dimensional Euclidean spaceforms. The case of spherical spaceforms was first set by Klein (1890) and Killing (1891). The problem was fully solved much later (Wolf, 1960). Eventually, the classification of homogeneous hyperbolic spaces was impulsed in the 1970's ; it is now an open field of intensive mathematical research (Thurston, 1979, 1997).

Going back to relativistic cosmology, the discovery of non-static solutions by Friedmann (1922) and, independently, Lemaître (1927), opened a new era for models of the universe as a whole (see, e.g., Luminet, 1997 for an epistemological analysis). Although Friedmann and Lemaître are generally considered as the discoverers of the big bang concept –at least of the notion of a dynamical universe evolving from an initial singularity –, one of their most original considerations, devoted to the topology of space, was overlooked. As they stated, the homogeneous isotropic universe models (F–L models) admit spherical, Euclidean or hyperbolic space sections according to the sign of their (constant) curvature (respectively positive, zero or negative). In addition, Friedmann (1923) pointed out the topological indeterminacy of the solutions in his popular book on general relativity, and he emphasized how the Einstein's theory was unable to deal with the global structure of spacetime. He gave the simple example of the cylinder – a locally Euclidean surface which has not the same topology as the plane. Generalizing the argument to higher dimensions, he concluded that several topological spaces could be used to describe the same solution of Einstein's equations.

Topological considerations were fully developed in his second cosmological article (Friedmann, 1924), although primarily devoted to the analysis of hyperbolic solutions. Friedmann clearly outlined the fundamental limitations of relativistic cosmology : “Without additional assumptions, the Einstein's world equations do not answer the question of the finiteness of our universe”, he wrote. Then he described how space could become finite (and multi-connected) by suitably identifying points. He also predicted the possible existence of “ghost” images of astronomical sources, since at the same point of a multi-connected space an object and its ghosts would coexist. He added that “a space with positive curvature is always finite”, but he recognized the fact that the mathematical knowledge of his time did not allow him to “solve the question of finiteness for a negatively-curved space”.

Comparing with Einstein's reasoning, it appears that the Russian cosmologist had no prejudice in favour of a simply-connected topology. Certainly Friedmann believed that only spaces with finite volume were physically realistic. Prior to his discovery of hyperbolic solutions, the cosmological solutions derived by Einstein, de Sitter and himself had a positive spatial curvature, thus a finite volume. With negatively-curved spaces, the situation became problematic, because the “natural” topology of hyperbolic space has an infinite volume. It is the reason why Friedmann, in order to justify the physical pertinence of his solutions, emphasized the possibility of compactifying space by suitable

identifications of points.

Lemaître fully shared the common belief in the closure of space. In a talk given at the Institut Catholique de Paris (Lemaître, 1978), the Belgian physicist expressed his view that Riemannian geometry “dissipated the nightmare of infinite space”. His two major cosmological models (the non-singular, “Eddington-Lemaître” model, 1927, and the singular, “hesitating universe” model, 1931) assumed positive space curvature. Thus Lemaître thoroughly discussed the possibility of elliptical space, that he preferred to the spherical one. Later, Lemaître (1958) also noticed the possibility of hyperbolic spaces as well as Euclidean spaces with finite volumes for describing the physical universe.

Such fruitful ideas of cosmic topology remained widely ignored by the main stream of big bang cosmology. Perhaps the Einstein-de Sitter model (1932), which assumed Euclidean space and eluded the topological question, had a negative influence on the development of the field. Almost all subsequent textbooks and monographies on relativistic cosmology assumed that the global structure of the universe was either the finite hypersphere, or the infinite Euclidean space, or the infinite hyperbolic space, without mentioning at all the topological indeterminacy. As a consequence, some confusion settled down about the real meaning of the terms “open” and “closed” used to characterize the F-L solutions, even in recent specialized articles (e.g. White and Scott, 1996). Whereas they apply correctly to time evolution (open models stand for ever-expanding universes, closed models stand for expanding-contracting solutions), they do not properly describe the extension of space (open for infinite, closed for finite). Nevertheless it is still frequent to read that the (closed) spherical model has a finite volume whereas the (open) Euclidean and hyperbolic models have infinite volumes. The correspondance is true only in the very special case of a simply-connected topology *and* a zero cosmological constant. According to Friedmann’s original remark, in order to know if a space is finite or infinite, it is not sufficient to determine the sign of its spatial curvature, or equivalently in a cosmological context to measure the ratio of the average density to the critical value : additional assumptions are necessary - those arising from topology, precisely.

Until 1995, investigations in cosmic topology were rather scarce (see references in LaLu95). From an epistemological point of view, it seems that the prejudice in favour of simply-connected (rather than multi-connected) spaces was of the same kind as the prejudice in favour of static (rather than dynamical) cosmologies during the 1920’s. At a first glance, an “economy principle” (often useful in physical modelling) could be invoked to preferably select the simply-connected topologies. However, on one hand, new approaches of spacetime, such as quantum cosmology, suggest that the smallest closed hyperbolic manifolds are favored (Cornish, Gibbons & Weeks, 1998), thus providing a new paradigm for what is the “simplest” manifold. On the other hand, present astronomical data indicate that the average density of the observable universe is less than the critical value ($\Omega = 0.3 - 0.4$), thus suggesting that we live in a negatively-curved F-L universe. Putting together these two requisites, cosmologists must face the fact that a negatively-curved space with a finite volume is necessarily multi-connected.

2 New developments

In the last decades, much effort in observational and theoretical cosmology has been directed towards determining the curvature of the universe. The problem of topology of spacetime was generally ignored within the framework of classical relativistic cosmology. It began to be seriously discussed in quantum gravity for various reasons : the spontaneous birth of the universe from quantum vacuum requires the universe to have compact space-like hypersurfaces, and the closure of space is a necessary condition to render tractable the integrals of quantum gravity (Atkatz & Pagels, 1982). However, the topology of spacetime also enters in a fundamental way in classical general relativity. Many cosmologists were surprisingly unaware of how topology and cosmology could fit together to provide new insights in universe models. Aimed to create a new interest in the field of cosmic topology, the extensive review by LaLu95 stressed on what multi-connectedness of the Universe would mean and on its observational consequences. However two different papers (Stevens et al, 1993 ; de Oliveira Costa & Smoot, 1995) declared that the small universe idea was “dead” ; in fact, drawing general conclusions from few examples mostly taken in the Euclidean case, they did not take into account the most interesting spaces for realistic universe models, namely the compact hyperbolic manifolds, which require a quite different treatment (LaLu95, Cornish et al., 1997a). Ironically enough, a worldwide interest for the subject has flourished since 1995, both from an observational point of view and from a theoretical one : approximately the same amount of papers in cosmic topology have been published within the last 3 years as in the previous 80 years ! Interesting progress has been achieved in mathematics as well as in cosmology. I briefly summarize below some of these advances.

2.1 Mathematical advances

Just remember here that the method for classifying the admissible topologies of a manifold is to determine the universal covering space and the fundamental polyhedron, and to calculate the holonomy group acting on the polyhedron. Particularly important cases for application to cosmology are the locally homogeneous and isotropic 3 – dimensional Riemannian manifolds, *i.e.* admitting one of the three geometries of constant curvature. The correspondance between the local homogeneous 3-geometries in Thurston’s sense and the Bianchi–Kantowski–Sachs classification of homogeneous cosmological models has been fully clarified (Rainer, 1996).

Any space of constant curvature M can be expressed as a quotient $M = \tilde{M}/\Gamma$, where the universal covering space \tilde{M} is either the Euclidean space \mathbf{R}^3 if $K = 0$, the hypersphere \mathbf{S}^3 if $K > 0$ or the hyperbolic 3-space \mathbf{H}^3 if $K < 0$, and Γ is a discrete subgroup of isometries without fixed point of \tilde{M} .

Classification of Euclidean and spherical spaceforms being achieved (see Wolf, 1984), only the case of three-dimensional homogeneous hyperbolic manifolds was recently investigated. The first thing to retain is that the 2-dimensional case does not give a good intuition of what can happen in higher dimensions. The Mostow theorem illustrates an essential difference between 2-dimensional hyperbolic geometry and higher dimensions : while a surface of genus ≥ 2 supports uncountably many non equivalent hyperbolic

Table 1: **The smallest known CHMs**

Name	Volume	r_-	r_+	l_{min}
WMF	0.9427	0.5192	0.7525	0.5846
Thurston	0.9814	0.5354	0.7485	0.5780
s556(-1,1)	1.0156	0.5276	0.7518	0.8317
m006(-1,2)	1.2637	0.5502	0.8373	0.5750
m188(-1,1)	1.2845	0.5335	0.9002	0.4804
v2030(1,1)	1.3956	0.5483	1.0361	0.3662
m015(4,1)	1.4124	0.5584	0.8941	0.7942
s718(1,1)	2.2726	0.6837	0.9692	0.3392
m120(-6,1)	3.1411	0.7269	1.2252	0.3140
s654(-3,1)	4.0855	0.7834	1.1918	0.3118
v2833(2,3)	5.0629	0.7967	1.3322	0.4860
v3509(4,3)	6.2392	0.9050	1.3013	0.3458

structures, for $n \geq 3$ a connected oriented n -dimensional manifold supports *at most one* hyperbolic structure. It follows that geometric invariants such as the volume, or the lengths of closed geodesics are also topological invariants, and the curvature radius is a characteristic length scale for topology. It is the reason why the tentative classification of CHMs is based on increasing volumes.

Each CHM topology has a specific volume measured in curvature radius units. The absolute lower bound for the volume of CHMs, given previously as $V_{min} = 0.000082$, has been raised to $V_{min} = 0.16668$ (Gabai et al., 1996). Fortunately it has little effect on cosmological applications. The reason is that the true lower bound is almost surely 0.942707 (Weeks, 1998), corresponding to the smallest CHM that is presently known (Weeks, 1985; Matveev and Fomenko, 1988). The new V_{min} bound represents an improvement in the techniques of the proof, not an increase in the expected size of the smallest hyperbolic manifold.

The WFM manifold leaves room for many topological lens effects, since the volume of the observable universe is about 200 times larger than the volume of WFM space (Costa & Fagundes, 1998). Indeed, many CHMs have geodesics shorter than the curvature radius, leaving room to fit a great many copies of a fundamental polyhedron within the horizon radius, even for manifolds of volume ~ 10 . The publicly available program SNAPPEA (Weeks) is specially useful to unveil the rich structure of CHMs. Table I summarizes some of the results (r_- is the radius of the largest sphere in the covering space which can be inscribed in the fundamental polyhedron, r_+ is the radius of the smallest sphere in the covering space in which the fundamental polyhedron can be inscribed, l_{min} is the length of the shortest geodesic).

2.2 Cosmological advances

The global topology of the universe can be tested by studying the 3-D distribution of discrete sources and the 2-D fluctuations in the Cosmic Microwave Background (CMB). The methods are all based on the search for “ghost images” predicted by Friedmann (1924), namely topological images of a same celestial object such as a galaxy, a cluster or a spot in the CMB (the term “ghost” can lead to a confusion in the sense that all the images are on the same foot of reality). Such topological images can appear in a multi-connected space a characteristic length scale of which is smaller than the size of the observable space, because light emitted by a distant source can reach an observer along several null geodesics. In the 1970’s, systematic 2-D observations of galaxies, undertaken at the 6-m Zelentchuk telescope under the supervision of Schvartsman, allowed to fix lower limits to the size of physical space as $500h^{-1}Mpc$ (see LaLu95 and references herein). A new observational test based on 3-D analysis of clusters separations, the so-called “cosmic crystallographic method”, has been proposed by Lehoucq, Lachièze-Rey & Luminet (1996), and further discussed in the literature (Fagundes & Gausmann, 1997; Roukema & Blanoeil, 1998). Other 3-D methods, using special quasars configurations (Roukema, 1996) or X-ray clusters (Roukema & Edge, 1997), have also been devised, see Roukema (1998) for a summary. However, the poorness of 3-D data presently limits the power of such methods.

Some authors (de Oliveira-Costa et al., 1996), still believing that an inflationary scenario necessarily leads to an Einstein-de Sitter universe, looked for constraints on topology with the CMB by investigating the compact Euclidean manifolds (CEM) only. They found that toroidal universe with rectangular cells (the simplest CEM, described as E_1 in LaLu95’s classification), with cell size smaller than $3000h^{-1}Mpc$ for a scale-invariant power spectrum, were ruled out as “interesting cosmological models”. However, as shown by Fagundes & Gausmann (1997), CEMs remain physically meaningful even if the size of their spatial sections is of the same order of magnitude as the radius of the observable horizon. Using the method of cosmic crystallography (Lehoucq et al., 1996) they performed simulations showing sharp peaks in the distribution of distances between topological images.

In any case, CHMs appear today as the most promising specimens for cosmology, both from theoretical and observational grounds. Topological signature using the “circles in the sky” method (Cornish et al., 1996b) is difficult to detect in COBE data, but it could be possible with the future MAP and/or PLANCK data.

In fact, full cosmological calculations in CHMs (e.g. simulations of CMB fluctuations, or possible Casimir-like effects in the early universe) are difficult; they require calculations of eigenmodes of the Laplace operator acting on the compact manifold. The problem is not solved. Cornish & Turok (1998) recently suggested a method working in 2-D, but the 3-D case could reveal untractable. Compactness renders the calculations more difficult due to “geodesic mixing”, namely chaotic behaviour of geodesic bundles (Cornish et al., 1996a). Some authors (Levin et al, 1997, Cornish et al., 1997) were able to perform calculations in the “horn topology”, an open hyperbolic space introduced by Sokoloff and Starobinsky (1976), but the essential job remains to be done.

Another underdeveloped promising field is the interface between topology and the early universe at high energy (although below the Planck scale). Uzan and Peter (1997)

showed that if space is multi-connected on scales now smaller than the horizon size, the topological defects such as strings, domain walls, ... expected from GUT to arise at phase transitions, were very unlikely to form.

In my opinion, a major breakthrough in the field of cosmic topology would be to relate the topological length scale L_T with the cosmological constant Λ . In an unified scheme with two fundamental lengths scales – the Planck scale l_P and the inverse square root of the cosmological constant Λ , a consistent theory of quantum gravity should be able to predict the most probable value of L_T in terms of l_P and $\Lambda^{-1/2}$. Preliminary calculations in 2-D gravity models can be performed to test the idea.

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